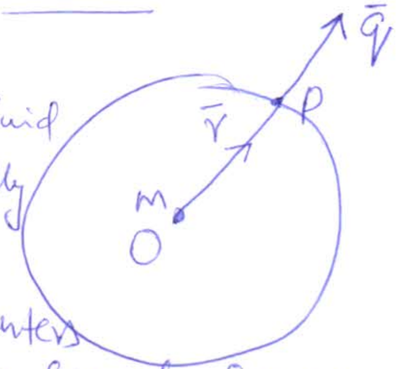
Lecture 1 MATH18-204 CHAPTER 4   Ref:- A TEXT BOOK OF FLUID DYNAMICS — FRANK CHORLTON, Pages 136–140

SOME THREE-DIMENSIONAL FLOWS

1. Sources, Sinks and Doublets



(a) Source:- Suppose at a point O in a fluid the flow is such that it is directed radially outwards from O in all directions and in a symmetrical manner. Then fluid enters the system through O which is termed a simple source.

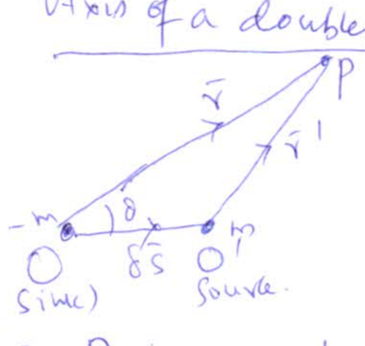
Defn:- Strength of the source:- If at O, the volume entering per unit time is $4\pi m$, where m is a constant, then 'm' is defined to be the strength of the source.

b) Sink:- Suppose that the flow is such that fluid is directed radially inwards to O from all directions in a symmetrical manner, then fluid leaves the system at O which is termed a simple sink.

RMK: A sink of strength m is a source of strength '$-m$' ($\because$ due to directions of source & sink)

(c) Doublet: A doublet is a combination of a source of strength 'm' and a sink of strength '$-m$' at a small distance 'δs' apart, where in the limit m is taken infinitely great and 'δs' infinitely small but so that the product '$m\,\delta s$' remains finite and equal to 'μ', is called a doublet strength.

Axis of a doublet: The line δs taken in the sense from '$-m$' to m (Sink to Source) is called the axis of a doublet.



2. Relations between the Strength of Source, Sink and Velocity

$$q = \frac{m}{r^2} \quad (\because \text{By Continuity Equation } 4\pi r^2 q = 4\pi m)$$

$\bar{q} = \frac{m}{r^2}\hat{r}$. Similarly, for the case of sink, we will find that $\boxed{\bar{q} = -\frac{m}{r^2}\hat{r}}$ (1)

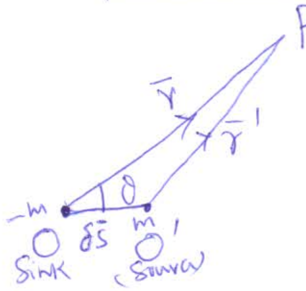
RMK: It is easy to check that $\text{Curl}\,\bar{q} = \bar{0}$ (except at $r=0$; By using the Curl in cylindrical Co-ordinates)

So, the flow is of the potential kind.

Let $\phi = \phi(r)$ be the velocity potential at P due to symmetry, we can see that $\nabla\phi = \phi'(r)\hat{r}$, $\phi'(r) = -m/r^2$

$\therefore \phi(r) = \frac{m}{r}$. (2)

3. To Find the velocity potential at P due to a doublet at O.



From this configuration shown here, assuming the flow to be entirely due to $-m$ at O and m at O', the velocity potential at P is $\phi = \phi_{(\text{at } O')} + \phi_{(\text{at } O)}$

$$= \frac{m}{r'} - \frac{m}{r} \quad \left(\text{by using Eqn (2)}\right)$$

$$\phi = \frac{m(r - r')}{r r'}$$

(On Rationalizing Newmerator)

$$\phi = \frac{m(r - r')(r + r')}{r r'(r + r')} = \frac{m(r^2 - r'^2)}{r r'(r + r')} = \frac{m(\bar{r} - \bar{r}')\cdot(\bar{r} + \bar{r}')}{r r'(r + r')}$$

$$= \frac{m\,\delta\bar{s}\cdot(\bar{r} + \bar{r}')}{r r'(r + r')} \quad \left(\because \bar{r} - \bar{r}' = \delta\bar{s}\right)$$

$$\phi = \frac{\bar{\mu}\cdot(\bar{r} + \bar{r}')}{r r'(r + r')}$$

Now keep O fixed and let $O' \to O$, so that $\bar{r}' \to \bar{r}$, μ staying constant.
Then in the limit we have $\phi = \frac{\bar{\mu}\cdot 2\bar{r}}{2r^3} = (\bar{\mu}\cdot\bar{r})\,r^{-3}$ — (3)

With $\angle POO' = \theta$, equation (3) reduces to $\phi = (\bar{\mu}\cdot\bar{r})\,r^{-2} = \bar{\mu}\cdot\left(\frac{\bar{r}}{|\bar{r}|}\right) r^{-2}$

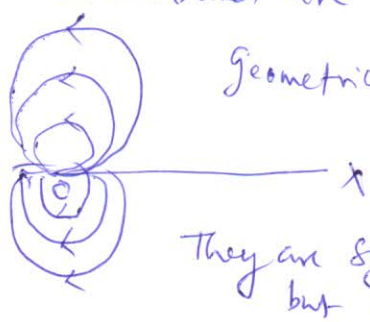
$$\therefore \boxed{\phi = (\bar{\mu}\cdot\hat{r})\,r^{-2} = \mu\, r^{-2}\cos\theta} \quad (\because \bar{\mu} = m\,\delta\bar{s})$$

Component of velocity can be found to be: $q_r = -\frac{\partial\phi}{\partial r} = +\frac{2\mu\cos\theta}{r^3}$

$$q_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{\mu\sin\theta}{r^3}$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\psi} = 0.$$

Streamlines are given by $\psi = \text{Const}$, $r = A\sin^2\theta$.



Geometrically, these stream lines lie in planes passing through the axis of the doublet.
They are symmetrical not only with respect to OX but also about the plane $\theta = \pi/2$.

RMK: The function $\varphi(r,\theta) = \frac{\mu\cos\theta}{r^2}$ shows that at points other than O must satisfy the Laplace Eqn in axisymmetric flow case

MMATH18-204: FLUID DYNAMICS Lecture 2:

Example 1

Doublet in a uniform stream

Ref: A Text Book of Fluid Dynamics – Frank Chorlton, Pages: 140 and 139

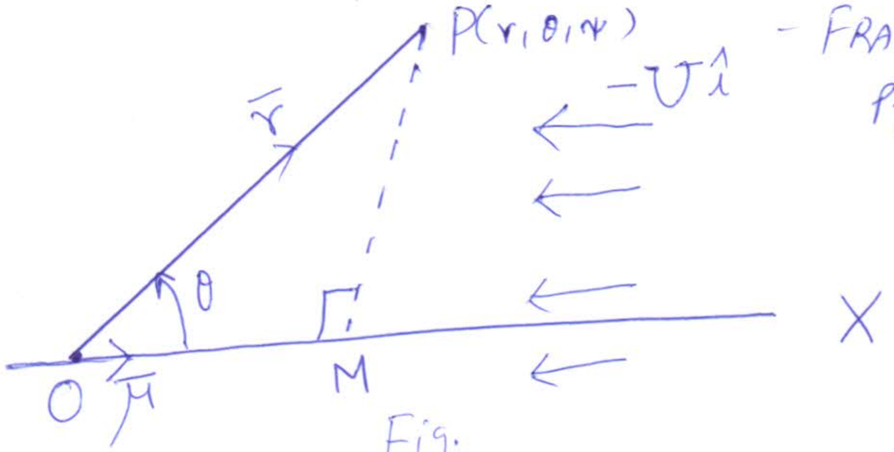


Fig.

This Fig. shows a doublet of vector moment $\bar{\mu} = \mu\hat{i}$ at O in a uniform stream whose velocity, in the absence of the doublet, would be $-U\hat{i}$. We first find the velocity potential $\phi(r, \theta)$ at the field point P in the fluid having spherical polar co-ordinates (r, θ, ψ). PM is the $\perp^r$ from P on OX and if $OM = x$, then the velocity potential at P due to the streamline is $Ux = Ur\cos\theta$. That due to the doublet at O is $\frac{\mu\cos\theta}{r^2}$ so that the total velocity potential at P is

$$\phi(r, \theta) = (Ur + \mu r^{-2})\cos\theta \longrightarrow (1)$$

The velocity components at P are $q_r = -\frac{\partial\phi}{\partial r} = -\left(U - \frac{2\mu}{r^3}\right)\cos\theta$ — (2)

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \left(U + \frac{\mu}{r^3}\right)\sin\theta; \quad q_\psi = \frac{-1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0.$$

We see that $q_r = 0$, whenever (i) $\cos\theta = 0$ i.e. $\theta = \pi/2$ and (ii) $r = \left(\frac{2\mu}{U}\right)^{1/3}$

Write $a = \left(\frac{2\mu}{U}\right)^{1/3}$, so that $\mu = \frac{1}{2}Ua^3$. Then it follows that there is no flow over the surface of the sphere $r = a$.

In fact for $r \geq a$, with $\mu = \frac{1}{2}Ua^3$ in equation (2), we obtain precisely the same velocity potential as was obtained for uniform flow past a stationary sphere of radius a. Thus, for the region $r \geq a$, the analysis of this sphere problem is the same as that of the dipole of strength $\frac{1}{2}Ua^3$ in the uniform stream of velocity $-U\hat{i}$, the axis of the dipole being the direction of $\hat{i}$.